Closed Expressions for Finite Transformations

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We find a solution to the following simple problem. Given an n x n matrix θ

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \cdot & \theta_{1n} \\ \theta_{21} & \theta_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta_{n1} & \cdot & \cdot & \cdot & \theta_{nn} \end{pmatrix} \quad (1)$$

where the θ_{ij} are complex numbers and where θ is diagonalizable, then find a closed expression for the exponentiated form $\mu = e^{i\theta}$ where

$$\mu = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots \quad (2)$$

By the Cayley-Hamilton theorm, the n x n matrix θ satisfies an n^{th} degree polynomial equation. Consequently, μ may be expressed in terms of a power series of degree n-1 in θ

$$\mu = \sum_{i=0}^{n-1} a_i \theta^i \quad (3)$$

The a_i are functions of the invariants of θ and the problem is to find these functions.

Now define the quantity

$$T_i \equiv tr\left(\mu\theta^i\right) \quad (4)$$

The trace is invarient under the transformation that diagonalizes θ and μ. Let the diagonal form of θ be $\overline{\theta}$ and the diagonal form of μ be $\overline{\mu}$, then

$$T_i = tr\left(\overline{\mu}\overline{\theta}^i\right) \tag{5}$$

Also, using (3) and (4) we have

$$T_i = \sum_{j=0}^{n-1} a_j tr\left(\theta^{i+j}\right) = \sum_{j=0}^{n-1} a_j tr\left(\overline{\theta}^{i+j}\right) \tag{6}$$

Let

$$A_{ij} \equiv tr\left(\overline{\theta}^{i+j}\right) \qquad i, j = 0,1,2...n-1 \tag{7}$$

Then

$$T_i = \sum_{j=0}^{n-1} A_{ij} a_j \qquad i = 0,1,2...n-1 \tag{8}$$

is a linear system of equations for a_j. Assuming that the determinant $|A| \neq 0$, the inverse to A exists and we have

$$a_i = \sum_{j=0}^{n-1} A_{ij}^{-1} T_j \tag{9}$$

As an example we find the 2 x 2 representation of SU_2. θ is hermitian and traceless. Consequently, $\overline{\theta}$ has the form

$$\overline{\theta} = \begin{pmatrix} \phi/2 & 0 \\ 0 & -\phi/2 \end{pmatrix} \tag{10}$$

where ϕ is real, $\overline{\mu}$ has the form

$$\bar{\mu} = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \tag{11}$$

Computing T_i and A_{ij} we get

$$T_0 = 2\cos\frac{\phi}{2} \qquad T_1 = \frac{\phi}{2}\sin\frac{\phi}{2} \tag{12}$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \phi^2/2 \end{pmatrix} \tag{13}$$

Inverting A we get

$$A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2/\phi^2 \end{pmatrix} \tag{14}$$

and solving for a_i from (9)

$$a_0 = \cos\frac{\phi}{2} \qquad a_1 = \frac{i}{\phi/2}\sin\frac{\phi}{2} \tag{15}$$

we arrive at the familiar form

$$e^{i\theta} = \cos\frac{\phi}{2} + \frac{i}{\phi/2}\sin\frac{\phi}{2}\theta \tag{16}$$

where ϕ is the angle of rotation.