# Random Walks in Space and Time

Daniel Lee Wenger

Abstract

Several models of multiple particle scattering in homogeneous matter are treated as random walks and analyzed with techniques of multi-variate probability theory.

The time is treated as a random variable equivalently to the space variables. This approach is in contrast to that of treating the time as a parameter as is done in the special theory of stochastic processes.

The transition of the time variable from a probability to a conditional variable is associated with a transition from Markovian to non-Markovian processes.

For exponentially distributed times of flight, exact spatial probability density functions and fluxes, conditional upon the time, are obtained for the cases of constant speed isotropic scattering in one and two dimensions.

Mathematically, the models may be considered as a mixture of random walk and renewal processes.

**(1)**

The mathematical theory of random walk processes dates from 1905 when Pearson posed the problem of determining the probability density of a particle, actually an inebriate, undergoing displacements of equal length in random directions.

The names of Rayleigh and Kluyver are associated with early solutions to this problem [1].

Markov expressed a general procedure for treating such problems. The method, which bears his name, has been generalized by Chandrasekhar[1] , and will be used here in a somewhat different form.

To develop familiarity with the methods used here, consider the following process.

A particle, starting at the origin at $t = 0$, undergoes a collision after a time $t_0$. The time $t_0$ is taken to be a random variable described by a probability density $\rho(t_0)$. In this paper we restrict ourselves to the exponential density function.

$$\rho(t_0) = \sigma e^{-\sigma v t_0} \qquad 0 \le t_0 \tag{1.1}$$

Here $v$ is the speed of the particle and $\sigma$ is a microscopic scattering cross-section, or, inverse mean free path

As a density function, $\rho(t_0)$ satisfies

$$\int_0^\infty dt_0 \rho(t_0) = 1 \qquad 0 \le \rho(t_0) \tag{1.2}$$

After the collision, the particle moves off in a new direction and undergoes a second collision after a time $t_1$. $t_1$, as well, is described by (1.1).

Continuing in this way, there are, after $n$ collision, $n+1$ random variables $t_i; \quad i = 0,1\cdots,n$. Note that the counting convention used here counts the number of collisions prior to the last collision.

The total elapsed time $t$ after $n$ collisions is

$$t = t_0 + t_1 + \cdots + t_n \tag{1.3}$$

a random variable which is a function of the $n+1$ statistically independent random variables $t_i$, The probability density for $t$, conditional upon $n$ collisions having occurred is

$$\rho(t\,|\,n) = \int_R dt_0 \int dt_1 \cdots \int dt_n \rho(t_0)\rho(t_1)\cdots\rho(t_n) \tag{1.4}$$

where the region of integration R is the surface in $n+1$ dimensional space defined by equation (1.3).

Using the Dirac delta function, (1.4) may be expressed as

$$\rho(t\,|\,n) = \int_0^\infty dt_0 \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \rho(t_0)\rho(t_1)\cdots\rho(t_n)\delta\left(t - \sum_{j=0}^{n} t_j\right) \tag{1.5}$$

where now the range of integration is over the entire domain of the variables $t_i$.

To evaluate (1.5), the delta function is expressed as an inverse Fourier transform.

$$\delta\left(t - \sum_{j=0}^{n} t_j\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_t e^{ik_t\left(t - \sum_{j=0}^{n} t_j\right)} \tag{1.6}$$

Substitution of (1.6) into (1.5), a change in the order of integration and the fact that the $\rho(t_i)$ are all the same, yields

$$\rho(t\,|\,n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_t e^{ik_t}\left[\int_0^\infty dt_0 \rho(t_0) e^{-ik_t t_0}\right]^{n+1} \tag{1.7}$$

Substitution of (1.1) and evaluation of the integral over $t_0$ gives

$$\rho(t\,|\,n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_t e^{ik_t t_0}\left[\frac{\sigma v}{\sigma v + ik_t}\right]^{n+1} \tag{1.8}$$

(1.8) is evaluated by the method of residues using a contour enclosing the upper half plane including the pole at $k_t = i\sigma v$. The result is

$$\begin{aligned} \rho(t\,|\,n) &= \sigma v \frac{(\sigma v t)^n}{n!} e^{-\sigma v t} \qquad 0 \le t \\ \rho(t\,|\,n) &= 0 \qquad t \le 0 \end{aligned} \tag{1.9}$$

(1.9) may be integrated to show that $\rho(t\,|\,n)$ is normalized.

$$\int_0^\infty \rho(t\,|\,n) = 1 \tag{1.10}$$

The method followed here is essentially the Markov method for evaluating the density function of a sum of random variables [1].

**(2)**

We turn now to the study of a scattering model in one dimension. Consider a particle that moves along the $x$ axis and undergoes scatters that reverse the particles direction without changing it's speed. The particle starts, say, at the origin moving in the positive $x$ direction.

After $n$ collisions the x coordinate of the particle is

$$x = vt_0 - vt_1 + vt_2 - \cdots + (-1)^n vt_n = v\sum_{j=0}^{n}(-1)^j vt_j \tag{2.1}$$

Using the methods of section (1), the joint probability density for $x$ and $t$ conditional upon $n$ collisions having occurred, is

$$\rho_+(x,t\,|\,n) = \int_0^\infty dt_0 \cdots \int_0^\infty dt_n \rho(t_0)\cdots\rho(t_n)\delta\left(t - \sum_{j=0}^{n} t_j\right)\delta\left(x - v\sum_{j=0}^{n}(-1)^j t_j\right) \tag{2.2}$$

where the + subscript indicates that the particle started in the positive direction.

The product of delta functions is a natural expression of multiple constraints and gives results in agreement with standard methods of constrained integrations [2].

Using (1.6) and

$$\delta\left(x - v\sum_{j=0}^{n}(-1)^j t_j\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_x e^{ik_x\left(x - v\sum_{j=0}^{n}(-1)^j t_j\right)} \tag{2.3}$$

we have, after a change in order of integration, grouping of similar integrals and evaluation of the integrals over $t_i$,

$$\rho_+(x,t\,|\,n) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_t e^{ik_t t}\int_{-\infty}^{\infty} dk_x e^{ik_x x}\left[\frac{\sigma v}{(\sigma v + ik_t + ivk_x)^p(\sigma v + ik_t - ivk_x)^q}\right]^{n+1} \tag{2.4}$$

Here $p + q = n + 1$ and $p = q$ or $p = q + 1$. $p$ is the number of translations in the $+x$ direction, $q$ is the number in the $-x$ direction.

New variables are defined implicitly by the equations

$$
\begin{aligned}
k_t &= \frac{\sigma v}{2}\left(k_+ + k_-\right) \\
k_x &= \frac{\sigma}{2}\left(k_+ - k_-\right)
\end{aligned}
\tag{2.5}
$$

The Jacobian of the transformation is $\frac{\sigma^2 v}{2}$. The effect of the change in variable is to separate the integrand, giving

$$
\rho(x,t\,|\,n) = \frac{\sigma^2 v}{8\pi^2}\int_{-\infty}^{\infty} dk_+ \frac{e^{ik_+\sigma(vt+x)/2}}{\left(1+ik_+\right)^p}\int_{-\infty}^{\infty} dk_- \frac{e^{ik_-\sigma(vt-x)/2}}{\left(1+ik_-\right)^q} \tag{2.6}
$$

(2.6) is evaluated, as before, by the method of residues.

For $n = 0$, that is, before a collision occurs, $p = 1$, $q = 0$, and

$$
\rho(x,t\,|\,0) = \sigma v e^{-\sigma v t}\delta(vt - x) \qquad 0 \le t \tag{2.7}
$$

For $n > 0$,

$$
\begin{aligned}
&\rho_+(x,t\,|\,n) = \sigma v e^{-\sigma v t}\left(\frac{\sigma}{2}\right)^n \frac{(vt+x)^{p-1}}{(p-1)!}\frac{(vt-x)^{q-1}}{(q-1)!} \\
&\rho_+(x,t\,|\,n) = 0 \qquad vt < |x| \\
&\qquad -vt \le x \le vt \qquad 0 \le t
\end{aligned}
\tag{2.8}
$$

(2.8) may be expressed, for $n$ even, as

$$
\rho_+(x,t\,|\,n) = \frac{\sigma^2 v}{2} e^{-\sigma v t}\left(\frac{\sigma}{2}\right)(vt+x)\frac{(\sigma s/2)^{n-2}}{\left(\frac{n}{2}\right)!\left(\frac{n-2}{2}\right)!} \tag{2.9}
$$

and for $n$ odd, as

$$
\begin{aligned}
\rho_+(x,t\,|\,n) &= \frac{\sigma^2 v}{2} e^{-\sigma v t}\frac{(\sigma s/2)^{n-1}}{\left(\frac{n}{2}\right)!\left(\frac{n-2}{2}\right)!} \\
s &= \sqrt{(vt)^2 - x^2}
\end{aligned}
\tag{2.10}
$$

It is seen from (2.7) that the density for the particle before a collision occurs is an exponentially damped delta function centered at the position of the particle as it moves in the positive x direction.

From (2.8), one sees that after a collision the particle might be found anywhere between $-vt \le x \le vt$, but with greater probability near the origin.

Integrating (2.8) over x recaptures (1.9).

$$\int_{-vt}^{vt} dx \rho_{+}(x,t \mid n) = \sigma v \frac{(\sigma vt)^{n}}{n!} e^{-\sigma vt} = \rho(t \mid n) \tag{2.11}$$

(Note that the integral over x for $\rho_{+}(x,t \mid 0)$ has limits of integration $\pm\infty$, since x is only restricted by the delta function, thus, the point $x = vt$ is always included).

**(3)**

The question now arises, may we somehow determine a probability density for x in which contributions from every n occur.

Until now, we have been considering continuous Markov series in the variable x and t, or alternatively, a discrete Markovian process in the variable n [3]. This is seen by expressing the density for $n+1$ scatters in terms of the density for $n$ scatters, and recognizing therein the Schmoluchowski-Chapman-Kolmogorov equation. ( This actually identifies the process as a member of a slightly larger set of processes including the exceptional non-Markovian processes that satisfy the S-C-K equation [3].

$$\rho(x,t \mid n+1) = \int dx' dt' dx_{0} dt_{0} \rho(x_{0} t_{0}) \delta(t - t_{0} - t') \delta(x - x_{0} - x') \tag{3.1}$$

This equation will be discussed in section (13).

The operations of this section will destroy the Markovian property and render the processes non-Markovian.

To proceed, form the conditional probability density $\rho(x \mid t,n)$

$$\rho(x \mid t,n) = \frac{\rho(x,t \mid n)}{\rho(t \mid n)} \qquad \rho(t \mid n) \neq 0 \tag{3.2}$$

Consider as well the probability $P(n \mid t)$, the probability of n scatters given the time t. The probability density $\rho(x,n \mid t)$ is given in terms of these new quantities by

$$\rho(x,n \mid t) = \rho(x \mid t,n) P(n \mid t) = \frac{\rho(x,t \mid n) P(n \mid t)}{\rho(t \mid n)} \tag{3.3}$$

$\rho(x,n \mid t)$ is a mixture of a density in x and a distribution in n.

We now evaluate $P(n\,|\,t)$. Consider the space of events labeled by a particular ordered set of numbers $(t_0, t_1, \cdots t_n, \cdots t_\infty,)$ where the $t_i$ satisfy the condition $\sum_{i=0}^{\infty} t_i = t$. The $t_i$ are weighted by the function $\rho(t_i)$.

The probability $P(n\,|\,t)$ is given by the ratio of the number of events in the subspace of events defined by $t_0 + t_1 + \ldots + t_n = t \quad t_{n+1}, t_{n+2}, \ldots\ t_\infty = 0$ to the total number of events in the space of events. The events common to more than one subspace cause no problem since they are of measure zero.

$P(n\,|\,t)$ is thus given by

$$P(n\,|\,t) = \frac{\int_0^\infty dt_0 \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \rho(t_0)\rho(t_1)\cdots\rho(t_n)\delta(t - t_0 - t_1 \cdots - t_n)}{\sum_{j=0}^{\infty} \int_0^\infty dt_0 \int_0^\infty dt_1 \cdots \int_0^\infty dt_j \rho(t_0)\rho(t_1)\cdots\rho(t_j)\delta(t - t_0 - t_1 \ldots - t_j)} \tag{3.4}$$

or,

$$P(n\,|\,t) = \frac{\rho(t\,|\,n)}{\sum_{j=0}^{\infty} \rho(t\,|\,j)} \tag{3.5}$$

where $\rho(t\,|\,n)$ is the probability density for a sum of $n+1$ random variables with density $\rho(t_i)$.

Substituting this result into (3.3) gives

$$\rho(x, n\,|\,t) = \frac{\rho(x, t\,|\,n)}{\sum_{j=0}^{\infty} \rho(t\,|\,j)} \tag{3.6}$$

Now sum over n to find

$$\rho(x\,|\,t) = \sum_{n=0}^{\infty} \rho(x, n\,|\,t) = \frac{\sum_{n=0}^{\infty} \rho(x, t\,|\,n)}{\sum_{j=0}^{\infty} \rho(t\,|\,j)} \tag{3.7}$$

This is a general result independent of the particular form of the densities. The only difficulty arises when the sum in the denominator of (3.5) is not finite.

For Exponentially distributed times of flight, $\rho(t \mid n)$, as given by (1.9), has the property that

$$\sum_{n=0}^{\infty} \rho(t \mid n) = \sigma v \tag{3.8}$$

consequently,

$$\rho(x, n \mid t) = \frac{1}{\sigma v} \rho(x, t \mid n) \tag{3.9}$$

and the probability density for x conditional upon t is

$$\rho(x \mid t) = \frac{1}{\sigma v} \sum_{n=0}^{\infty} \rho(x, t \mid n) \tag{3.10}$$

$P(n \mid t)$ is recognized as the Poisson distribution. The expected value of n, conditional upon t, is given by

$$\langle n \mid t \rangle = \sum_{n=0}^{\infty} n P(n \mid t) = \sigma v t \tag{3.11}$$

The expected value of n is thus proportional to t, a result in contrast to the approach of taking n proportional to t as was often done in early random walks.

A word here is appropriate regarding the historical development of probability theory. Many results are available with respect to the properties of chained processes and in particular of Markov chains.

In relating these processes to physical processes, it has been common to make the assumption that temporal development is given by the chain number. The primary consequence is that the processes continue to be Markovian in the continuous variable t.

Here, another notion of temporal development is indicated by equation (3.6). These processes are in general non-Markovian.

The pseudo-Poisson process [3] is a special case of (3.6) and is characterized by

$$\rho(x, t \mid n) = \rho(x \mid n) \rho(t \mid n) \tag{3.12}$$

Here $\rho(t \mid n)$ is given by (1.9) and statistical independence of x and t is explicitly indicated.

**(4)**

We now use the results (2.7) and (2.8) for one dimensional back-scatter and formula (3.10) to derive a probability density for x, conditional upon t, and in which contributions from all n occur, namely

$$\rho_+(x,t\,|\,n) = e^{-\sigma v t}\left\{\delta(vt-x) + \sum_{n=1}^{\infty}(\sigma/2)^n \frac{(vt+x)^{p-1}}{(p-1)!}\frac{(vt-x)^{q-1}}{(q-1)!}\right\} \qquad -vt \le x \le vt \qquad 0 \le t \tag{4.1}$$

where $p+q=n+1$ and $p=q$ or $p=q+1$.

By grouping odd and even terms, (4.1) takes the form

$$\rho_+(x\,|\,t) = e^{-\sigma v t}\left\{\delta(vt-x) + \frac{\sigma}{2}\sum_{k=0}^{\infty}\frac{(\sigma s/2)^{2k}}{k!^2} + \frac{\sigma^2}{4}(vt+x)\sum_{k=0}^{\infty}\frac{(\sigma s/2)^{2k}}{k!(k+1)!}\right\} \qquad s=\sqrt{(vt)^2-x^2} \qquad -vt \le x \le vt \qquad 0 \le t \tag{4.2}$$

The two series in (4.2) are identified as modified Bessel functions [4], giving the result,

$$\rho_+(x\,|\,t) = e^{-\sigma v t}\left\{\delta(vt-x) + \frac{\sigma}{2}I_0(\sigma s) + \frac{\sigma}{2}\frac{(vt+x)}{s}I_1(\sigma s)\right\} \qquad s=\sqrt{(vt)^2-x^2} \qquad -vt \le x \le vt \qquad 0 \le t \tag{4.3}$$

With the substitution $s = vt\sin\theta$, $x = vt\cos\theta$, (4.2) may be integrated to show

$$\int_{-vt}^{vt} dx\,\rho_+(x\,|\,t) = 1 \tag{4.4}$$

The above probability density is that for a particle starting in the $+x$ direction. The density $\rho_-(x\,|\,t)$ for a particle starting in the $-x$ direction is

$$\rho_-(x\,|\,t) = \rho_+(-x\,|\,t) \tag{4.5}$$

The density for a particle starting isotropically is

$$\rho_{iso}(x\,|\,t) = \frac{1}{2}\rho_-(x\,|\,t) + \frac{1}{2}\rho_+(x\,|\,t) \tag{4.6}$$

or,

$$\rho_{iso}(x\,|\,t) = \frac{1}{2}e^{-\sigma vt}\left\{\delta(vt-x)+\delta(vt+x)+\sigma I_0(\sigma s)+\sigma\frac{vt}{s}I_1(\sigma s)\right\} \tag{4.7}$$

Here $\sigma$ is the cross-section for back-scatter.

**(5)**

We now treat the one dimensional isotropic scattering problem. In this mode, the particle, after each scatter, is found, with equal probability, to be moving in the $+x$ or the $-x$ direction.

In the previous case, there were $n+1$ independent random variables $t_i; i = 0,1,2,\cdots n$. Here an additional $n+1$ random variables are introduced, namely, $x_i; i = 0,1,2,\cdots n$. $x_i$ is the displacement between the ith collision and the (i+1)th collision.

We are interested in the two dependent variables

$$t = t_0 + t_1 + \cdots t_n \tag{5.1}$$

and

$$x = x_0 + x_1 + \cdots x_n \tag{5.2}$$

But now, $x_i$ and $t_i$ are not statistically independent and the joint probability density for $x_i$ and $t_i$ is introduced. We choose

$$\rho(x_i,t_i) = \sigma v e^{-\sigma vt_i}\left\{\frac{1}{2}\delta(vt_i-x_i)+\frac{1}{2}\delta(vt_i+x_i)\right\} \tag{5.3}$$

Integrating (5.3) over $x_i$ returns the exponential density for $t_i$.

$$\int_{-\infty}^{\infty}dx_i\rho(x_i,t_i) = \sigma v e^{-\sigma vt_i} = \rho(t_i) \tag{5.4}$$

Here $\sigma$ is the total isotropic scattering cross-section.

Using the techniques of section (2), $\rho(x,t\,|\,n)$ is given by

$$\rho(x,t\,|\,n) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_t e^{ik_t t}\int_{-\infty}^{\infty}dk_x e^{ik_x x}\left[\int_0^{\infty}dt_i\int_{-\infty}^{\infty}dx_i e^{-i(k_t t_i + k_x x_i)}\rho(x_i,t_i)\right]^{n+1} \tag{5.5}$$

or

$$\rho(x,t\,|\,n)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_t e^{ik_t t}\int_{-\infty}^{\infty}dk_x e^{ik_x x}\left[\frac{\sigma v(\sigma v+ik_t)}{(\sigma v+ik_t)^2+(k_x v)^2}\right]^{n+1} \tag{5.6}$$

The change of variable (2.5) leads to

$$\rho(x,t\,|\,n)=\frac{\sigma^2 v}{2}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_+\int_{-\infty}^{\infty}dk_- e^{ik_+\sigma(vt+x)/2}e^{ik_-\sigma(vt-x)/2}\left[\frac{(1+ik_+)+(1+ik_-)}{2(1+ik_+)(1+ik_-)}\right]^{n+1} \tag{5.7}$$

Expansion of the numerator in a binomial series provides a separation of variables and the integrations are then performed as in (2.6). The result is, for $n=0$,

$$\rho(x,t\,|\,0)=\frac{\sigma v}{2}e^{-\sigma vt}\left\{\delta(vt-x)+\delta(vt+x)\right\} \tag{5.8}$$

and for $n>0$,

$$\begin{aligned}\rho(x,t\,|\,n)&=\frac{\sigma v}{2}\frac{(\sigma vt/2)^n}{n!}e^{-\sigma vt}\left\{\delta(vt-x)+\delta(vt+x)\right\}\\&+\frac{\sigma v}{2}\left(\frac{\sigma}{4}\right)^n e^{-\sigma vt}\sum_{j=1}^{n}\binom{n+1}{j}\frac{(vt-x)^{n-j}}{(n-j)!}\frac{(vt+x)^{j-1}}{(j-1)!}\\&-vt\le x\le vt \qquad 0\le t\end{aligned} \tag{5.9}$$

(5.8) is the unscattered portion of the density, composed of a part representing the particle moving in the $+x$ direction and a part representing the particle moving in the $-x$ direction, both parts decaying exponentially with time.

(5.9) represents the particle after it has undergone n collisions. The delta function portion is the forward scattered component of the density.

Making use of formula (3.10), $\rho(x\,|\,t)$ takes the form

$$\begin{aligned}\rho(x\,|\,t)&=\frac{1}{2}e^{-\sigma vt}\left\{\delta(vt-x)+\delta(vt+x)\right\}\\&+\frac{1}{2}e^{-\sigma vt}\sum_{n=1}^{\infty}\left(\frac{\sigma}{4}\right)^n\sum_{j=1}^{n}\binom{n+1}{j}\frac{(vt-x)^{n-j}}{(n-j)!}\frac{(vt+x)^{j-1}}{(j-1)!}\end{aligned} \tag{5.10}$$

Now label the cross-section occurring in (5.10) as $\sigma_{iso}$ for isotropic scattering, and introduce $\sigma_b$, the cross-section for back scattering, by

$$\sigma_b=\frac{1}{2}\sigma_{iso} \tag{5.11}$$

(4.7) and (5.10) may now be used to establish an interesting identity. There is but one fundamental process, with two equivalent descriptions. One is that if a collision occurs with probability $\sigma_b$, a scatter occurs in the reverse direction with probability 1. The other view is that given a collision with a probability $\sigma_{iso}$, the probability of back scatter is 1/2.

The delta function terms are clearly equivalent. Equating the remaining portions of (4.7) and (5.10), the following identity is found.

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\binom{n+1}{j}\frac{(a)^{n-j}}{(n-j)!}\frac{(b)^{j-1}}{(j-1)!} &= e^{a+b}\left\{2\sum_{k=0}^{\infty}\frac{(ab)^k}{k!k!}+\sum_{k=0}^{\infty}\frac{(a+b)(ab)^k}{k!(k+1)!}\right\} \\ &= e^{a+b}\left\{2I_0\left(2\sqrt{ab}\right)+\frac{a+b}{\sqrt{ab}}I_1\left(2\sqrt{ab}\right)\right\}\end{aligned} \tag{5.12}$$

where a and b are arbitrary complex numbers.

The final result for $\rho(x\,|\,t)$ for one dimensional isotropic scattering is

$$\begin{gathered}\rho(x\,|\,t)=\frac{1}{2}e^{-\sigma vt/2}\left\{\delta(vt-x)+\delta(vt+x)+\frac{\sigma}{2}I_0(\sigma s/2)+\frac{\sigma}{2}\frac{vt}{s}I_1(\sigma s/2)\right\} \\ s=\sqrt{(vt)^2-x^2} \qquad -vt\le x\le vt \qquad 0\le t\end{gathered} \tag{5.13}$$

where $\sigma$ is the cross-section for isotropic scattering.

**(6)**

We now consider a two dimensional scattering process. The particle is constrained to move in the two dimensional plane and undergoes isotropic scattering at each collision. The dependent random variables of interest are

$$\begin{aligned}t &= t_0+t_1+\cdots+t_n \\ x &= x_0+x_1+\cdots+x_n \\ y &= y_0+y_1+\cdots+y_n\end{aligned} \tag{6.1}$$

The three variables $t_i, x_i, y_i$ are statistically interdependent. This is expressed via the joint probability density

$$\rho\left(x_i,y_i,t_i\right)=\frac{\sigma e^{-\sigma vt_i}}{2\pi t_i}\delta\left(vt_i-\sqrt{x_i^{\,2}+y_i^{\,2}}\right) \tag{6.2}$$

$\sigma$ is the total isotropic scattering cross-section.

(6.2) has the property of returning the exponential density for $t_i$ when integrated over $x_i$ and $y_i$.

The density for the three variables of (6.1) is

$$\rho(x,y,t\,|\,n)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}dk_t e^{ik_t t}\int_{-\infty}^{\infty}dk_x e^{ik_x x}\int_{-\infty}^{\infty}dk_y e^{ik_y y} \cdot\left[\int_{\varepsilon}^{\infty}dt_i e^{-ik_t t_i}\int_{-\infty}^{\infty}dx_i e^{-ik_x x_i}\int_{-\infty}^{\infty}dy_i e^{-ik_y y_i}\rho(x_i,y_i,t_i)\right]^{n+1} \tag{6.3}$$

A change of variable from $x_i$ and $y_i$ to polar coordinates, where $k_x x_i + k_y y_i = kr_i\cos\theta$ and $k=\sqrt{k_x{}^2+k_y{}^2}$, allows the quantity in brackets to be evaluated.

$$\begin{aligned}\frac{\sigma}{2\pi}\int_{\varepsilon}^{\infty}dt_i e^{-k_t t_i}\frac{e^{-\sigma v t_i}}{t_i}\int_0^{2\pi}d\theta\int_0^{\infty}dr_i r_i e^{-ikr_i\cos\theta}\delta(vt_i-r_i)&=\sigma v\int_0^{\infty}dt_i e^{-(ik_t+\sigma v)t_i}J_0(kvt_i)\\&=\frac{\sigma v}{\left[(\sigma v+ik_t)^2+(kv)^2\right]^{1/2}}\end{aligned} \tag{6.4}$$

or,

$$\rho(x,y,t\,|\,n)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}dk_t e^{ik_t t}\int_{-\infty}^{\infty}dk_x e^{ik_x x}\int_{-\infty}^{\infty}dk_y e^{ik_y y} \cdot\left[\frac{\sigma v}{\left[(\sigma v+ik_t)^2+(kv)^2\right]^{1/2}}\right]^{n+1} \tag{6.5}$$

After a similar change of variable for $k_x$ and $k_y$, namely, $k_x x + k_y y = kr\cos\theta$ and $r=\sqrt{x^2+y^2}$, (6.5) takes the form

$$\rho(x,y,t\,|\,n)=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}dk_t e^{ik_t t}\int_0^{\infty}dkJ_0(kr)\left[\frac{\sigma v}{\left[(\sigma v+ik_t)^2+(kv)^2\right]^{1/2}}\right]^{n+1} \tag{6.6}$$

The Hankel transform occurring in (6.6) is converted to a Fourier transform using the identity

$$\int_0^{\infty}dk\,kJ_0(rk)\frac{1}{\left(s^2+k^2\right)^{z+1}}=\frac{\Gamma(z+1/2)}{\sqrt{\pi}\,\Gamma(z+1)}\int_0^{\infty}dk\frac{\cos(kr)}{\left(s^2+k^2\right)^{z+1/2}} \tag{6.7}$$

a relationship valid for $\mathrm{Re}\,z>-1/2$, Re $s>0$, which translates into $n>1$.

This identity is proved in the appendix and leads to

$$\rho(x,y,t\,|\,n)=\frac{\sigma}{(2\pi)^2}\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}\int_{-\infty}^{\infty}dk_t e^{ik_t t}\int_{-\infty}^{\infty}dk e^{ikr}\left[\frac{\sigma v}{\left[(\sigma v+ik_t)^2+(kv)^2\right]}\right]^{\frac{n+1}{2}} \tag{6.8}$$

A change in variables

$$k_t=\frac{\sigma v}{2}(k_+ + k_-)$$
$$k=\frac{\sigma}{2}(k_+ - k_-) \tag{6.9}$$

allows the integrations to be performed, with the results, for $n=0$,

$$\rho(x,y,t\,|\,0)=\frac{\sigma v}{2\pi r}e^{-\sigma vt}\delta(vt-r)$$
$$0<r \qquad 0<t \tag{6.10}$$

and for $n>0$,

$$\rho(x,y,t\,|\,n)=\frac{\sigma^2 v}{2\pi s}e^{-\sigma vt}\frac{(\sigma s)^{n-1}}{(n-1)!}$$
$$s=\sqrt{(vt)^2-r^2} \qquad r=\sqrt{x^2+y^2} \tag{6.11}$$
$$0\le r\le vt \qquad 0\le t$$

Here, $\rho(x,y,t\,|\,1)$ has been determined by a direct integration.

Summing over n and dividing by $\sigma v$ gives

$$\rho(x,y\,|\,t)=\frac{1}{2\pi}e^{-\sigma vt}\left\{\frac{1}{r}\delta(vt-r)+\frac{\sigma}{s}e^{\sigma s}\right\} \tag{6.12}$$

The density for the variable r is

$$\rho(r\,|\,t)=e^{-\sigma vt}\left\{\frac{1}{r}\delta(vt-r)+\frac{\sigma r}{s}e^{\sigma s}\right\}$$
$$s=\sqrt{(vt)^2-r^2} \qquad 0\le r\le vt \qquad 0\le t \tag{6.13}$$

It is a simple integration to show

$$\int_0^{vt} dr\, \rho(r\,|\,t) = 1 \qquad (6\ .14)$$

Without giving the details, the density for measuring x alone is

$$\begin{aligned}\rho(x\,|\,t) &= \int_{-\sqrt{(vt)^2-x^2}}^{\sqrt{(vt)^2-x^2}} dy\,\rho(x,y\,|\,t)\\ &= e^{-\sigma vt}\left\{\frac{1}{\pi s}+\frac{\sigma}{2}I_0(\sigma s)+\frac{\sigma}{2}L_0(\sigma s)\right\} \qquad (6.15)\\ s &= \sqrt{(vt)^2 - x^2} \qquad -vt \le r \le vt \qquad 0 \le t\end{aligned}$$

**(7)**

In this section we compute the two dimensional flux density $\rho(x,y,t,\mu\,|\,n)$.

We take $\mu$ to be the cosine of the angle between the vector $(x,y)$ and the flux vector. $\rho(x,y,t,\mu\,|\,0)$ and $\rho(x,y,t,\mu\,|\,1)$ are computed directly and $\rho(x,y,t,\mu\,|\,n)$ for $n>1$ is then computed using Fourier representations of the delta functions.

$$\begin{aligned}\rho(x,y,t,\mu\,|\,0) &= \int dx_0 \int dy_0 \int dt_0\\ &\cdot \delta(x-x_0)\delta(y-y_0)\delta(t-t_0)\\ &\cdot \frac{\sigma}{2\pi t_0} e^{-\sigma v t_o}\delta\left(vt_0 - \sqrt{x_0^{\,2}+y_0^{\,2}}\right) \qquad (7.1)\\ &\cdot \delta\left(\mu - \frac{xx_0+yy_0}{\sqrt{x^2+y^2}\sqrt{x_0^{\,2}+y_0^{\,2}}}\right)\\ &= \frac{\sigma}{2\pi t}e^{-\sigma vt}\delta\left(vt-\sqrt{x^2+y^2}\right)\delta(\mu-1) \qquad 0<t\end{aligned}$$

For $\rho(x,y,t,\mu\,|\,1)$,

$$\begin{aligned}\rho(x,y,t,\mu\,|\,1) &= \int dx_0 \int dx_1 \int dy_0 \int dy_1 \int dt_0 \int dt_1\\ &\cdot \frac{\sigma}{2\pi t_0} e^{-\sigma v t_o}\delta\left(vt_0 - \sqrt{x_0^{\,2}+y_0^{\,2}}\right)\\ &\cdot \frac{\sigma}{2\pi t_1} e^{-\sigma v t_1}\delta\left(vt_1 - \sqrt{x_1^{\,2}+y_1^{\,2}}\right) \qquad (7.2)\\ &\cdot \delta(x-x_0-x_1)\delta(y-y_0-y_1)\delta(t-t_0-t_1)\\ &\cdot \delta\left(\mu - \frac{xx_1+yy_1}{\sqrt{x^2+y^2}\sqrt{x_1^{\,2}+y_1^{\,2}}}\right)\end{aligned}$$

The integrations over $x_0, y_0$ and $t_0$ are performed first.

After making the substitutions $x_1 = r'\cos\theta, y_1 = r'\sin\theta$ and performing the integrations over $r'$, we get

$$\rho(x,y,t,\mu\,|\,1) = \left(\frac{\sigma v}{2\pi}\right)^2 e^{-\sigma vt}\int_0^{2\pi} d\theta \int_0^t dt_1 \frac{\delta(t-t_1) - \sqrt{r^2 + (vt_1)^2 - 2vt_1 r\mu}}{v(t-t_1)} \cdot \delta\left(\mu - \frac{x\cos\theta + y\sin\theta}{r}\right) \tag{7.3}$$

Using the relationship

$$\int dx\, g(x)\, \delta(f(x)) = \sum_{zeros\ of\ f(x)} \frac{g(x)}{f'(x)}\bigg|_{zero\ of\ f(x)} \tag{7.4}$$

there follows,

$$\rho(x,y,t,\mu\,|\,1) = \frac{\sigma^2 v}{2\pi^2}\frac{e^{-\sigma vt}}{\sqrt{1-\mu^2}\,(vt-\mu r)} \tag{7.5}$$

Now computing $\rho(x,y,t,\mu\,|\,n)$ for $n > 1$,

$$\begin{aligned}\rho(x,y,t,\mu\,|\,n) = {} & \frac{1}{(2\pi)^4}\left(\frac{\sigma}{2\pi}\right)^{n+1}\int_{-\infty}^{\infty} dk_x e^{ik_x x}\int_{-\infty}^{\infty} dk_y e^{ik_y y}\int_{-\infty}^{\infty} dk_t e^{ik_t t}\int_{-\infty}^{\infty} dk_\mu e^{ik_\mu \mu} \\ & \cdot\left[\int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 \int_{\varepsilon}^{\infty} dt_0 \frac{e^{-\sigma vt_0}}{t_0}\delta\left(vt_0 - \sqrt{{x_0}^2 + {y_0}^2}\right)e^{-i\left(k_x x_0 + k_y y_0 + k_t t_0\right)}\right]^n \\ & \cdot\int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dy_n \int_{\varepsilon}^{\infty} dt_n \frac{e^{-\sigma vt_n}}{t_n}\delta\left(vt_n - \sqrt{{x_n}^2 + {y_n}^2}\right) \\ & \cdot e^{-i\left(k_x x_n + k_y y_n + k_t t_n\right)} e^{-ik_\mu \frac{xx_n + yy_n}{\sqrt{x^2+y^2}\sqrt{{x_n}^2+{y_n}^2}}}\end{aligned} \tag{7.6}$$

The substitutions

$$\begin{aligned} x_n &= r_n \cos\theta_n \\ y_n &= r_n \sin\theta_n \\ dx_n dy_n &= r_n dr_n d\theta_n \end{aligned}$$

and the relationship

$$\int_0^{2\pi} d\theta e^{i(a\cos\theta + b\sin\theta)} = 2\pi J_0\left(\sqrt{a^2+b^2}\right) \tag{7.7}$$

leads to

$$\rho(x,y,t,\mu \mid n) = \frac{\sigma v}{2\pi^4}\int_{-\infty}^{\infty} dk_x e^{ik_x x}\int_{-\infty}^{\infty} dk_y e^{ik_y y}\int_{-\infty}^{\infty} dk_t e^{ik_t t}\int_{-\infty}^{\infty} dk_\mu e^{ik_\mu \mu}$$
$$\cdot\left[\sigma v\int_0^{\infty} dt_0 e^{-(\sigma v + ik_t)t_0} J_0(kvt_0)\right]^n$$
$$\cdot\int_0^{\infty} dt_n e^{-(\sigma v + ik_t)t_n} J_0\left(\sqrt{a^2+b^2}\right) \tag{7.8}$$
$$a^2+b^2 = k^2v^2t_n{}^2 + 2vt_nk_\mu\frac{(k_x x + k_y y)}{r} + k\mu^2$$
$$k^2 = k_x{}^2 + k_y{}^2 \qquad r^2 = x^2+y^2$$

Introducing $k_x x + k_y y = kr\cos\alpha$ and using the known Laplace transform of $J_0$, we get

$$\rho(x,y,t,\mu \mid n) = \frac{\sigma v}{2\pi^4}\int_{-\infty}^{\infty} dk_t e^{ik_t t}\int_{-\infty}^{\infty} dk_\mu e^{ik_\mu \mu}\int_0^{2\pi} d\alpha \int_0^{\infty} dk\, k e^{ikr\cos\alpha}$$
$$\cdot\left[\frac{\sigma v}{\sqrt{(\sigma v + ik_t)^2 + (kv)^2}}\right]^n \tag{7.9}$$
$$\cdot\int_0^{\infty} dt_n e^{-(\sigma v + ik_t)t_n} J_0\left(\sqrt{(kvt_n)^2 + 2vt_nk\cos\alpha + k_\mu{}^2}\right)$$

Neumann's expansion theorem gives

$$J_0\left(\sqrt{(kvt_n)^2 + 2vt_nk\cos\alpha + k_\mu{}^2}\right) = \sum_{j=-\infty}^{\infty} J_j(-kvt_n)J_j(k\mu)\cos(j\alpha) \tag{7.10}$$

which allows the $k_\mu$ and $\alpha$ integrations to be performed, namely

$$\int_{-\infty}^{\infty} dk_\mu e^{ik_\mu \mu} J_j(k_\mu) = \frac{2i^j T_j(\mu)}{\sqrt{1-\mu^2}} \tag{7.11}$$

and

$$\int_0^{2\pi} d\alpha e^{ikr\cos\alpha}\cos(j\alpha) = 2\pi i^j J_j(kr) \tag{712}$$

Again using Neumann's theorem

$$\sum_{j=-\infty}^{\infty}(-1)^j J_j(-kvt_n)J_j(kr)T_j(\mu) = J_0\left(k\sqrt{(vt_n)^2 + r^2 - 2vt_n r\mu}\right) \tag{7.13}$$

we get

$$\begin{aligned}\rho(x,y,t,\mu \mid n) &= \frac{2\sigma v}{(2\pi)^3}\frac{1}{\sqrt{1-\mu^2}}\int_0^\infty dk_t e^{ik_t t}\int_0^\infty dt_n e^{-(\sigma v + ik_t)t_n} \\ &\cdot\int_0^\infty dk\, k\left[\frac{\sigma v}{\sqrt{(\sigma v + ik_t)^2 + (kv)^2}}\right]^n J_0\left(k\sqrt{(vt_n)^2 + r^2 - 2vt_n r\mu}\right)\end{aligned} \tag{7.14}$$

The Hankel transform occurring in (7.14) is converted to a Fourier transform using the identity (6.7). This leads to

$$\begin{aligned}\rho(x,y,t,\mu \mid n) &= \frac{\sigma^{n+1}v}{\sqrt{\pi}(2\pi)^3}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{\sqrt{1-\mu^2}}\int_0^\infty dt_n e^{-\sigma v t_n} \\ &\cdot\int_0^\infty dk_t e^{ik_t(t-t_n)}\int_{-\infty}^\infty dk \frac{e^{ikw}}{\left[(\sigma + ik_t/v)^2 + k^2\right]^{\frac{n-1}{2}}} \\ w &= \sqrt{(vt_n)^2 + r^2 - 2vt_n r\mu}\end{aligned} \tag{7.15}$$

The transformation (6.9) gives

$$\begin{aligned}\rho(x,y,t,\mu \mid n) &= \frac{(\sigma^2 v)^2}{2\sqrt{\pi}(2\pi)^3}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{\sqrt{1-\mu^2}}\int_0^\infty dt_n e^{-\sigma v t_n} \\ &\cdot\int_{-\infty}^\infty dk_+ \frac{e^{ik_+\sigma[v(t-t_n)+w]/2}}{(1+ik_+)^{\frac{n-1}{2}}}\int_{-\infty}^\infty dk_- \frac{e^{ik_-\sigma[v(t-t_n)-w]/2}}{(1+ik_-)^{\frac{n-1}{2}}}\end{aligned} \tag{7.16}$$

The $k_+$ and $k_-$ integrations are now performed giving

$$\rho(x,y,t,\mu\,|\,n)=\frac{\left(\sigma^2 v\right)^2}{4\pi\sqrt{\pi}}\frac{e^{-\sigma vt}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\frac{1}{\sqrt{1-\mu^2}}\int_0^{t'}dt_n\left(\sigma s'/2\right)^{n-3}$$
$$s'^2=(vt)^2-r^2-2v(vt-r\mu)t_n \qquad (7.17$$
$$t'=\frac{(vt)^2-r^2}{2v(vt-r\mu)}$$

The integration may be performed and with

$$\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)=\frac{(n-2)!\sqrt{\pi}}{2^{n-2}} \qquad 1<n \qquad (7.18)$$

the following result is obtained.

$$\rho(x,y,t,\mu\,|\,n)=\frac{\sigma^2 v}{2\pi^2}\frac{e^{-\sigma vt}}{(vt-r\mu)\sqrt{1-\mu^2}}\frac{(\sigma s)^{n-1}}{(n-1)!}$$
$$s=\sqrt{(vt)^2-r^2} \qquad r^2=x^2+y^2 \qquad (7.19)$$
$$0\le r\le vt \qquad 0<t \qquad -1\le\mu\le 1 \qquad 0<n$$

Integration of $\rho(x,y,t,\mu\,|\,n)$ over $\mu$ gives

$$\int_{-1}^{1}d\mu\;\rho(x,y,t,\mu\,|\,n)=\rho(x,y,t\,|\,n) \qquad (7.20)$$

namely, the two dimensional density (6.11).

The flux density for $r$ due to all collisions, expressed in terms of $\theta$ the angle between the direction of the flux and the position vector, is given by

$$\rho(r,\theta\,|\,t)=e^{-\sigma vt}\left\{\delta(vt-r)\delta(\theta)+\frac{\sigma}{\pi}\frac{e^{\sigma s}}{(vt-r\cos\theta)}\right\}$$
$$s=\sqrt{(vt)^2-r^2} \qquad (7.21)$$
$$0\le r\le vt \qquad 0<t \qquad 0\le\theta\le\pi$$

This is truly a simple and elegant result.

## Appendix

The identity (6.7)

$$\Gamma(z+1)\int_0^\infty dk\frac{kJ_0(rk)}{\left(s^2+k^2\right)^{z+1}}=\frac{\Gamma(z+1/2)}{\sqrt{\pi}}\int_0^\infty dk\frac{\cos(kr)}{\left(s^2+k^2\right)^{z+1/2}} \qquad \text{(a.1)}$$

is now proved.

Express the gamma function in terms of its defining form. From the left side of (a.1),

$$\Gamma(z+1)\int_0^\infty dk\frac{kJ_0(rk)}{\left(s^2+k^2\right)^{z+1}}=\int_0^\infty dk\int_0^\infty dt\ t^z e^{-t}\frac{kJ_0(rk)}{\left(s^2+k^2\right)^{z+1}} \qquad \text{(a.2)}$$

Introduce the variable $u$ by the equation

$$u=\frac{t}{\left(s^2+k^2\right)} \qquad \text{(a.3)}$$

Equation (a.2) becomes

$$\Gamma(z+1)\int_0^\infty dk\frac{kJ_0(rk)}{\left(s^2+k^2\right)^{z+1}}=\int_0^\infty du\ u^z e^{-us^2}\int_0^\infty dke^{-uk^2}kJ_0(rk) \qquad \text{(a.4)}$$

The right side of (a.1) becomes

$$\frac{\Gamma(z+1/2)}{\sqrt{\pi}}\int_0^\infty dk\frac{\cos(kr)}{\left(s^2+k^2\right)^{z+1/2}}=\frac{1}{\sqrt{\pi}}\int_0^\infty du\ u^z e^{-us^2}\int_0^\infty dke^{-uk^2}\cos(kr) \qquad \text{(a.5)}$$

The problem reduces to one of showing that the relation

$$\int_0^\infty dke^{-uk^2}kJ_0(kr)=\frac{1}{\sqrt{\pi}\sqrt{u}}\int_0^\infty dke^{-uk^2}\cos(kr) \qquad \text{(a.6)}$$

is true.

The identity is established by expanding the integrands in series and integrating term by term. The left hand integral is

$$\int_0^\infty dke^{-uk^2}kJ_0(kr)=\frac{1}{2u}e^{-\frac{r^2}{4u}} \qquad \text{(a.7)}$$

and the right hand integral is

$$\frac{1}{\sqrt{\pi}\sqrt{u}}\int_0^\infty dke^{-uk^2}\cos(kr) = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{u}}e^{-\frac{r^2}{4u}} \tag{a.8}$$

The above proof is a rephrasing of certain results due to Poisson and Sonine [5].

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[5] G. Watson, "A Treatise on the Theory of Bessel Functions", Cambridge Press, 1962 (see Sections 6.32 and 13.6)